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QUASILINEAR RATE-TYPE CONSTITUTIVE EQUATIONS AND INCREMENTAL STRESS WAVES

I. SULICIU,† L. E. MALVERN and N. CRISTESCU‡

Center for Dynamic Plasticity, Department of Engineering Science, Mechanics, and Aerospace Engineering, University of Florida, Gainesville, Florida 32611

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Abstract—The requirement that the leading edge of an incremental loading pulse superimposed on a preloading that has reached a constant state will propagate at the elastic wave speed imposes some restrictions on the material functions describing a quasilinear rate-type material. These restrictions are discussed, and it is shown that the incremental wave behavior implies the existence of at least one relaxation boundary, and that a natural continuous transition from the quasilinear equation to a semilinear equation occurs in the neighborhood of the relaxation boundary.

How the knowledge of a relaxation boundary can be used in determining the material functions is also discussed.

1. INTRODUCTION

A rate-type constitutive equation exhibiting both time-independent and time-dependent plasticity has been shown by Cristescu[1] to give a better account of a group of finite-amplitude plastic wave propagation experiments than either a rate-independent constitutive equation or the semilinear rate-type equation of Sokolovskii[2, 3] and Malvern[4–6]. The quasilinear equation does not in general predict the observed incremental plastic wave speed in a bar preloaded either quasistatically[7, 8] or dynamically[9, 10] in such a way that the strain–time curve has reached a plateau or is increasing very slowly when the incremental pulse is applied. The experiments show that the leading edge of the incremental pulse propagates at the elastic bar-wave speed $c_0 = (E/\rho_0)^{1/2}$, as predicted by the semilinear constitutive equation, and not at the plastic wave-speed of the rate-independent theory.

The incremental wave behavior can be incorporated into the rate-type equation by introducing a coefficient function with discontinuous dependence on stress and strain, so that the function is replaced by the constant elastic modulus E whenever: (1) stress is decreasing, or (2) the stress–strain point is near a specified "relaxation boundary" in the stress–strain plane. Cristescu[1] has used such a discontinuous coefficient function.

The purpose of the research reported here was to determine if the observed incremental wave behavior could be predicted by a quasilinear rate-type constitutive equation with continuous coefficient functions (constitutive assumption I, section 2), and, if so, to determine the restrictions on the continuous coefficient functions, $\phi(\varepsilon, \sigma)$ and $\psi(\varepsilon, \sigma)$ of equation (2), imposed by the requirement that the leading edge of an incremental wave travel at the

† On leave from the Institute of Mathematics, Bucharest, Romania.
‡ On leave from the University of Bucharest, Romania.

elastic wave speed when it comes after a plateau has been reached in the preloading strain history (constitutive assumption II, section 3).

As a consequence, a natural connection between the quasilinear model and the semilinear model is obtained. The observed behavior implies the existence of a certain curve (relaxation boundary) in the stress–strain plane, such that, in the neighborhood of this curve, $\phi(\varepsilon, \sigma) = E$ and $\psi(\varepsilon, \sigma) = 0$. See theorem 3.1 in section 3. In section 4, examples will be discussed based on expansion of the solution of equation (2) in power series and retention of only a few terms.

2. PRELIMINARIES

We consider one-dimensional motion of a rod or a motion which can be satisfactorily described using a single position coordinate [X is the initial position (Lagrangian or material coordinate) and x is the displaced position (Eulerian or spatial coordinate)], and a single component σ (force per initial unit area—positive in compression) and $\varepsilon(= 1 - (\partial x(X, t)/\partial X))$ of stress and strain tensors, respectively.

Constitutive assumption I′

One supposes that, for every fixed section X of the rod, there exists a three-dimensional bounded domain D_3 and two functions $\phi', \psi' : D_3 \to R$ so that

$$\dot{\sigma} = \phi'(t, \varepsilon, \sigma)\dot{\varepsilon} + \psi'(t, \varepsilon, \sigma). \quad (1)$$

In what follows a more restrictive hypothesis will be used.

Constitutive assumption I

One supposes that, for every fixed section X of the rod, there exists a plane bounded domain D and two functions $\phi, \psi : D \to R$ so that

$$\dot{\sigma} = \phi(\varepsilon, \sigma)\dot{\varepsilon} + \psi(\varepsilon, \sigma). \quad (2)$$

In other words, the constitutive assumption I (or I′) asserts that, in any fixed section X of the rod, the increment $\Delta\sigma(t)$ is determined by increments $\Delta\varepsilon(t)$ and Δt and by the values of the material functions ϕ and ψ of the state at the time t (if at time t, $\varepsilon(t)$ and $\sigma(t)$ are known). The notation $D \to R$ represents a mapping by the functions ϕ and ψ of the domain D of the ε, σ-plane into the real numbers. D is assumed to be bounded and to contain the origin $\varepsilon = 0$, $\sigma = 0$, since equation (2) can describe actual material behavior only for bounded values of ε and σ.

Definition 2.1

A material satisfying constitutive assumption I (or I′) is called a rate-type material of the first order (see Truesdell and Noll[11]). If the material functions ϕ and ψ do not depend explicitly on the material particle (X coordinate), then the material is called homogeneous. If ϕ = constant, then the material is called semilinear rate-type material (Sokolovskii[2, 3], Malvern[4, 5]). If $\phi \neq$ constant, then the material is a quasilinear rate-type material (Malvern [6], Cristescu[12, 13], Lubliner[14]). For more detailed discussions and references, see for example, Cristescu[15].

We present briefly some results from[16] that will be needed in our argument (see also [17]). In[16] sufficient continuity and smoothness conditions on the functions ϕ and ψ are given to establish the following results. For continuously differentiable strain history $\varepsilon(t)$ it is sufficient to establish the main result, theorem 3.1, that ϕ be continuously differentiable and ψ be continuous. We consider first the equation

$$\dot{\sigma} = \phi(\varepsilon, \sigma)\dot{\varepsilon}. \tag{2'}$$

Let $\varepsilon(t)$ be a function with the following properties $\varepsilon \in C^1_{[t_0, T]}$ and $\dot{\varepsilon}(t) \neq 0$, $\varepsilon(t_0) = \varepsilon_0$. Let $(\varepsilon_0, \sigma_0) \in D$ be an arbitrary point. Then we may write (2′) as

$$\frac{\partial \sigma}{\partial \varepsilon} = \phi(\varepsilon, \sigma) \qquad \sigma(\varepsilon_0) = \sigma_0. \tag{3}$$

Solutions of this equation will be useful for constructing solutions of equation (2) by variation of parameters.

Equation (3) has a global unique solution

$$\left.\begin{aligned} &\sigma = f(\varepsilon, \varepsilon_0, \sigma_0) \\ &\bar{\omega}_- = \tilde{\omega}_-(\varepsilon_0, \sigma_0) < \varepsilon < \tilde{\omega}_+(\varepsilon_0, \sigma_0) = \bar{\omega}_+ \\ &\left(\bar{\omega}_\pm, \lim_{\varepsilon \to \omega \pm} f(\varepsilon, \varepsilon_0, \sigma_0)\right) \in \partial D \end{aligned}\right\} \tag{4}$$

where ∂D denotes the boundary of D. The points

$$(\varepsilon, f(\varepsilon, \varepsilon_0, \sigma_0)) \in D \qquad \varepsilon \in (\bar{\omega}_-, \bar{\omega}_+)$$

form a smooth curve through $(\varepsilon_0, \sigma_0)$, while $\bar{\omega}_-$ and $\bar{\omega}_+$ denote the values of ε at the intersection of the curve with ∂D on the left and right, respectively.

Now, we keep ε_0 fixed and put $\sigma_0 = \tau(t)$. We determine the function τ so that the solutions (4) of equation (3) satisfy equation (2). In fact, we apply here the Lagrange method of parameter variation. We have

$$\dot{\sigma} = \frac{\partial f}{\partial \varepsilon}\dot{\varepsilon} + \frac{\partial f}{\partial \tau}\dot{\tau} = \phi(\varepsilon, f(\varepsilon, \varepsilon_0, \tau))\dot{\varepsilon} + \psi(\varepsilon, \varepsilon_0, \tau)).$$

Therefore, using equation (3), we obtain the following equation for τ,

$$\frac{\partial f}{\partial \tau}(\varepsilon, \varepsilon_0, \tau)\dot{\tau} = \psi(\varepsilon, f(\varepsilon, \varepsilon_0, \tau)), \qquad \tau(t_0) = \sigma_0. \tag{5}$$

From this equation $\tau(t)$ results as a functional of $\varepsilon(\cdot)$ on the interval $[t_0, t) \subset [t_0, T]$ for $t_0 \leq t < \omega_D(\varepsilon(\cdot), \sigma_0) \leq T$. Here ω_D is the largest t from $[t_0, T]$ such that $(\varepsilon(t), \sigma(t)) \in D$ for $t \in [t_0, \omega_D)$, where $\sigma(t) = f(\varepsilon(t), \varepsilon_0, \tau(t))$. (From the physical point of view ω_D is the last t from $[t_0, T]$ such that a material whose strain history is $\varepsilon(t)$ still responds according to the rate-type constitutive equation.) The functionals τ and ω_D are uniquely determined by the history of the strain and the shape of domain D (see[16]).

We have, now, the following result: The general solution of equation (2) is of the form

$$\sigma = f(\varepsilon, \varepsilon_0, \tau) \tag{6}$$

where f is determined by equation (3) and τ is determined by equation (5). We shall call τ the history parameter.

In[16] this result is given in a larger class of functions $\varepsilon = \varepsilon(t)$, including possible shock-wave solutions. There are given also the conditions under which the map $\varepsilon \to f(\varepsilon, \varepsilon_0, \tau)$ is a continuous Fréchet differentiable mapping.

3. THE CONTINUOUS TRANSITION FROM THE QUASILINEAR CONSTITUTIVE EQUATION TO THE SEMILINEAR EQUATION

Constitutive assumption II

One supposes that for any strain history $\tilde{\varepsilon}(t)$ of the form

$$\tilde{\varepsilon}(t) = \begin{cases} \varepsilon_*(t) & t \in [t_0, t_1] \\ \varepsilon_*(t_1) = \varepsilon_* = \text{const.} & t \in (t_1, T], \end{cases}$$

where $\varepsilon_*(t)$ is an arbitrary nondecreasing continuous function, there exists a $t_2 \in (t_1, T]$ such that the leading edge of any incremental pulse superposed over $\tilde{\varepsilon}(t)$ at a time $t_3 \geq t_2$ will propagate with the elastic bar-wave speed $c_0 = [E/\rho_0]^{1/2}$.

Definition 3.1

We say that a rate-type material possesses a natural rest configuration if $\varepsilon(t) \equiv 0$ for any $t \in (T_1, T_2)$ and $\sigma(T_1) = 0$ imply $\sigma(t) \equiv 0$ for $t \in (T_1, T_2)$, for any $T_1 < T_2$, T_1, $T_2 \in R$.

Definition 3.2

A smooth arc curve $\sigma = g(\varepsilon)$ in the plane $\sigma - \varepsilon$ will be called a relaxation curve or a relaxation boundary (see Cristescu[1]) if along this curve $\psi(\varepsilon, g(\varepsilon)) = 0$.

Theorem 3.1

If ψ is continuous and ϕ is continuously differentiable, and (a) $\partial\phi/\partial\sigma \neq 0$, and (b) constitutive assumption II holds; then there exists a unique relaxation boundary, and in its neighborhood the propagation velocity is the elastic one.

Proof

For the sake of simplicity we suppose that a strain of the form $\tilde{\varepsilon}(t)$ from constitutive assumption II was applied to a rate-type material which previously was in a natural rest configuration and we choose $t_0 = 0$ (i.e. $\varepsilon(0) = 0$, $\sigma(0) = 0$). Then, due to the smoothness property of $\tau(t)$ (see[16]), the propagation velocity is given by

$$\rho_0 C_0{}^2 = \frac{\partial f}{\partial \varepsilon}(\varepsilon(t), \tau(t)) = \phi(\varepsilon(t), f(\varepsilon(t), \tau(t))). \tag{7}$$

For $\varepsilon(t) = \tilde{\varepsilon}(t)$, constitutive assumption II yields

$$\rho_0 C_0{}^2 = \frac{\partial f}{\partial \varepsilon}(\varepsilon_*, \tau(t)) = E, \qquad t \geq t_2. \tag{8}$$

On the other hand, using hypothesis (a) of the theorem, we get

$$\frac{\partial}{\partial \tau}\frac{\partial f}{\partial \varepsilon}(\varepsilon, \tau) = \frac{\partial}{\partial \tau}\phi(\varepsilon, f(\varepsilon, \tau)) = \frac{\partial \phi}{\partial \sigma}\frac{\partial f}{\partial \tau} \neq 0. \tag{9}$$

Since $\partial f/\partial \tau > 0$ (for proof, see[16]), it follows from (9) that (8) is invertible with respect to τ, and

$$\tau(t) = \tau_{\varepsilon_*} = \text{const.} \qquad t \geq t_2. \tag{10}$$

Therefore, we have the following result: for any strain history which reaches a plateau ε_* at time $t_1 > 0$, there exists a unique τ_{ε_*} (reached at time t_2—depending on strain history up to time t_1). This means that we have found a map

$$\varepsilon_* \xrightarrow{h} \tau_{\varepsilon_*}, \qquad \tau = h(\varepsilon). \tag{11}$$

From now on we shall omit subscript asterisk when there is no possibility of confusion. Introducing (11) in (6), we find

$$\sigma = f(\varepsilon, h(\varepsilon)) = g(\varepsilon). \tag{12}$$

The fact that (12) is a relaxation boundary follows from (10) and (5), i.e.

$$\psi(\varepsilon, g(\varepsilon)) = \frac{\partial f}{\partial \tau}(\varepsilon, h(\varepsilon))\dot{\tau} = 0. \tag{13}$$

The theorem is proved.

Therefore, constitutive assumption II leads to the conclusion that stress σ and history parameter τ possess a plateau in time together with ε. The theorem itself shows a continuous passing from the quasilinear material to the semilinear one. Thus, the semilinear model may be applied when we have a deformation process not too far from a relaxation boundary.

We have, also, the following relations for the functions g and h.

$$\begin{aligned} \frac{\mathrm{d}\sigma}{\mathrm{d}\varepsilon} &= \frac{\mathrm{d}g}{\mathrm{d}\varepsilon} = -\frac{\dfrac{\partial \phi}{\partial \varepsilon}(\varepsilon, g(\varepsilon))}{\dfrac{\partial \phi}{\partial \sigma}(\varepsilon, g(\varepsilon))} \\ \frac{\mathrm{d}\tau}{\mathrm{d}\varepsilon} &= \frac{\mathrm{d}h}{\mathrm{d}\varepsilon} = -\frac{\dfrac{\partial \phi}{\partial \varepsilon}(\varepsilon, g(\varepsilon))}{\dfrac{\partial \phi}{\partial \sigma}(\varepsilon, g(\varepsilon))\dfrac{\partial f}{\partial \tau}(\varepsilon, h(\varepsilon))} - \frac{E}{\dfrac{\partial f}{\partial \tau}(\varepsilon, h(\varepsilon))}. \end{aligned} \tag{14}$$

These relations show that if ϕ does not depend on ε, then $g = \text{const}$. Since we have previously assumed $\partial\phi/\partial\sigma \neq 0$, we assume that ϕ depends on both σ and ε in order to prove the theorem for a relaxation boundary $\sigma = g(\varepsilon)$ with nonconstant g. On the other hand, if the material has in the neighborhood of the point ($\varepsilon = 0$, $\sigma = 0$), linear elastic properties, i.e. $\phi = E$ in this neighborhood and yield point $(\varepsilon_y, \sigma_y)$ is the point where ϕ starts to be different from E, then this point appears as a singular point in theorem 3.1, and the material might have more than one relaxation boundary.

From the relations (14), we can see that if

$$0 \leq \frac{\mathrm{d}g}{\mathrm{d}\varepsilon} \leq E \qquad \varepsilon \in (\varepsilon_1, \varepsilon_2),$$

then, since $\partial f/\partial\tau > 0$, it follows that

$$\frac{\mathrm{d}h}{\mathrm{d}\varepsilon} \leq 0, \qquad \varepsilon \in (\varepsilon_1, \varepsilon_2).$$

Theorem 3.1 was established under the assumption that the strain history $\tilde{\varepsilon}(t)$ had reached an absolute plateau before the incremental pulse was applied. However, if we suppose in constitutive assumption II that $\tilde{\varepsilon}(t)$ is nearly a constant on the interval $(t_1, t_2]$, then, because of smoothness of all the functions, the conclusion of theorem 3.1 applies in an approximative manner.

4. APPLICATIONS

(a) Suppose that the function ϕ in equation (2) is "good enough" in order to develop the solution (6) in a power series. We shall choose also $\varepsilon_0 = \sigma_0 = 0$.

We take

$$\sigma = f(\varepsilon, \tau) = E\varepsilon + \tau + B_1\varepsilon^2 + B_2\varepsilon\tau + B_3\tau^2 + C_1\varepsilon^3 + C_2\varepsilon^2\tau + C_3\varepsilon\tau^2 + C_4\tau^3 + \dots \quad (15)$$

There, the coefficient of τ is chosen to be 1, which does not affect the generality.

(a_1) Consider now the first approximation

$$\sigma = f_1(\varepsilon, \tau) = E\varepsilon + \tau. \quad (16)$$

Then, by equation (7),

$$\frac{\partial f_1(\varepsilon, \tau)}{\partial \varepsilon} = E = \phi(\varepsilon, \sigma).$$

Therefore, the first approximation (equation (16)) leads to a general semilinear rate-type material (see Malvern[4, 5]).

The constitutive assumption II is satisfied automatically. Theorem 3.1 cannot be applied because $\partial\phi/\partial\sigma = 0$.

Equation (5) for τ becomes

$$\dot{\tau} = \psi(\varepsilon, E\varepsilon + \tau).$$

This equation, for $\varepsilon(t) = \bar{\varepsilon}(t)$, $t > t_1$ will give a plateau for $\tau(t)$ (and for $\sigma(t)$) only if some additional conditions are imposed on function ψ (see[17]).

(a_2) Let us consider now the third approximation

$$\sigma = f_3(\varepsilon, \tau) = E\varepsilon + \tau + B_1\varepsilon^2 + B_2\varepsilon\tau + B_3\tau^2 + C_1\varepsilon^3 + C_2\varepsilon^2\tau + C_3\tau^2\varepsilon + C_4\tau^3. \quad (17)$$

Applying to it condition (8), we get for $t \geq t_2$

$$2B_1\varepsilon + B_2\tau + 3C_1\varepsilon^2 + 2C_2\varepsilon\tau + C_3\tau^2 = 0.$$

This yields two roots $\tau_{1,2}$

$$\tau_{1,2} = -\frac{B_2 + 2C_2\varepsilon}{2C_3} \pm \frac{1}{2C_3}\sqrt{[p(\varepsilon)]} = h_3(\varepsilon). \quad (18)$$

From (17) and (18), we obtain†

$$\sigma = f_3(\varepsilon, h_3(\varepsilon)) = g_3(\varepsilon) = R(\varepsilon) \pm \frac{1}{2C_3} Q(\varepsilon)\sqrt{[p(\varepsilon)]} \quad (19)$$

where

$$p(\varepsilon) = B_2^2 + 4(B_2C_2 - 2B_1C_3)\varepsilon + 4(C_2^2 - 3C_1C_3)\varepsilon^2 \quad (20)$$

$$R(\varepsilon) = -\frac{B_2}{2C_3}\left(1 - \frac{B_2B_3}{C_3} + \frac{C_4B_2^2}{C_3^2}\right) + \left[E - \frac{2B_1B_3}{C_3} + \frac{3B_1B_2C_4}{C_3^2} + \frac{2B_2B_3C_2}{C_3^2} - \frac{3B_2^2C_2C_4}{C_3^3} - \frac{C_2}{C_3}\right]\varepsilon$$
$$+ \left[-B_1 - \frac{3B_3C_1}{C_3} + \frac{6B_1C_2C_4}{C_3^2} + \frac{B_2C_2}{2C_3} + \frac{9B_2C_1C_4}{2C_3^2} - \frac{6B_2C_2^2C_4}{C_3^3} + \frac{2B_3C_2^2}{C_3^2}\right]\varepsilon^2 + \left[-2C_1 + \frac{C_2^2}{C_3} - \frac{4C_2^2C_4}{C_3^3} + \frac{9C_1C_2C_4}{C_3^2}\right]\varepsilon^3 \quad (21)$$

† The nonuniqueness of the relaxation boundary described by (19) does not contradict theorem 3.1, since the hypothesis $\partial\phi/\partial\sigma \neq 0$ of the theorem has not been imposed on this example.

$$Q(\varepsilon) = 1 - \frac{B_2 B_3}{C_3} + \frac{B_2{}^2 C_4}{C_3{}^2} + \left[-\frac{2B_3 C_2}{C_3} - \frac{2B_1 C_4}{C_3} + \frac{4B_2 C_2 C_4}{C_3{}^2} \right]\varepsilon + \left[-C_2 - \frac{3C_1 C_4}{C_3} + \frac{4C_2{}^2 C_4}{C_3{}^2} \right]\varepsilon^2. \quad (22)$$

We now ask whether one of the two curves (19) can approximate the curve $\sigma = \beta\sqrt{\varepsilon}$ for $\varepsilon \in (\varepsilon_1, \varepsilon_2)$, since the constitutive equation

$$\sigma = \beta\sqrt{\varepsilon} \quad (23)$$

for aluminum has been proposed and justified experimentally by Bell[18]. It is suggested, by reloading experiments [10], that (23) would be relaxation boundary. Cristescu and Bell[19] have shown, by numerical computation, that a slightly modified form of equation (23), i.e.

$$\sigma = a + \beta\sqrt{(\varepsilon + \varepsilon_b)} \quad (24)$$

used as a finite constitutive equation, gives results in good agreement with experimental data on free flying impact of two bars.

First we put

$$g_3(\varepsilon) \equiv \beta\sqrt{\varepsilon} \qquad \varepsilon \in (\varepsilon_1, \varepsilon_2).$$

The resulting relations for B_i and C_i are incompatible.

Therefore, Bell's parabola, used as an exact relaxation boundary and expansion (17) disagree. We obtain the same negative result if we set

$$g_3(\varepsilon) \equiv a + \beta\sqrt{(\varepsilon + \varepsilon_b)} \qquad \varepsilon \in (\varepsilon_1, \varepsilon_2),$$

where a, ε_b, β are constants.

Of course, we could choose the coefficients B_i, C_i in (19) so that $g_3(\varepsilon)$ would be approximately equal to $\beta\sqrt{\varepsilon}$ on some interval, but it seems that it cannot be done on such a long interval as $(10^{-3}, 7 \times 10^{-2})$ for $\beta = 5{\cdot}6 \times 10^4$ (see Bell[20]).

(b) Suppose now, that, instead of a power series, we can develop $f(\varepsilon, \tau)$ in a series of the following form including fractional powers.

$$\sigma = f(\varepsilon, \tau) = A\varepsilon^{1/2} + \tau + E\varepsilon + E_1\varepsilon^{1/2}\tau + E_2\tau^2 + C_1\varepsilon^{3/2} + C_2\varepsilon\tau + C_3\varepsilon^{1/2}\tau^2 + C_4\tau^3 + D_1\varepsilon^2 + D_2\varepsilon^{3/2} + D_3\varepsilon\tau^2 + D_4\varepsilon^{1/2}\tau^3 + D_5\tau^4 + \dots \quad (25)$$

(b_1) The first approximation

$$\sigma = f_1(\varepsilon, \tau) = A\varepsilon^{1/2} + \tau. \quad (26)$$

cannot satisfy condition (8).

(b_2) Condition (8) applied to the second approximation

$$\sigma = f_2(\varepsilon, \tau) = A\varepsilon^{1/2} + \tau + E\varepsilon + E_1\varepsilon^{1/2}\tau + E_2\tau^2 \quad (27)$$

gives

$$\frac{\partial f_2}{\partial \varepsilon} = \frac{1}{2}A\tilde{\varepsilon}^{-1/2} + E + \frac{1}{2}E_1\tilde{\varepsilon}^{-1/2}\tau = E \qquad t \geq t_2.$$

This approximation leads to a general Malvern semilinear rate-type constitutive equation, as in (a_1), if we take

$$A = 0 \tag{28}$$

$$E_1 = 0 \tag{29}$$

where now, however, the expression $\tau + E_2\tau^2$ plays the role played by τ in (a) (compare, for example, equation (16) and (27)). Let

$$\tau_1 = \tau + E_2\tau^2. \tag{30}$$

Now it is τ_1 that must be negative if $\psi(\sigma, \varepsilon) \leq 0$, because $\dot{\tau}_1(t) = \psi(\tau_1 + E\varepsilon, \varepsilon) \leq 0$ and $\tau_1(0) = 0$. The change from ε to $\varepsilon^{1/2}$ in the series induces the replacement for τ given by equation (30).

(b_3) The third approximation is

$$\sigma = f_3(\varepsilon, \tau) = E\varepsilon + \tau + E_2\tau^2 + C_1\varepsilon^{3/2} + C_2\varepsilon\tau + C_3\varepsilon^{1/2}\tau^2 + C_4\tau^3. \tag{31}$$

Applying the same procedure as before, we find

$$\frac{\partial f_3}{\partial \varepsilon} = E + \frac{3}{2}C_1\tilde{\varepsilon}^{1/2} + C_2\tau + \frac{1}{2}C_3\tilde{\varepsilon}^{-1/2}\tau^2 \equiv E \qquad t \geq t_2 \tag{32}$$

and

$$\tau_1 = h_3(\varepsilon) = \tau + E_2\tau^2,$$

where

$$\begin{aligned} \tau = l_3(\varepsilon) &= \frac{[-C_2 \pm (C_2{}^2 - 3C_1C_3)^{1/2}]}{C_3}\varepsilon^{1/2} \\ &= \alpha^{\pm}\varepsilon^{1/2}, \qquad \alpha^{\pm} = [-C_2 \pm (C_2{}^2 - 3C_1C_3)^{1/2}]/C_3 \\ \sigma = g_3(\varepsilon) &= \alpha^{\pm}\varepsilon^{1/2} + (E + \alpha^{\pm 2}E_2)\varepsilon + (C_1 + C_2\alpha^{\pm} + C_3\alpha^{\pm 2} + C_4\alpha^{\pm 3})\varepsilon^{3/2}. \end{aligned} \tag{33}$$

The relaxation boundary given by formulae (33) can approximate the curve $\sigma = \beta\sqrt{\varepsilon}$, for a large interval for ε, if we choose

$$\begin{gathered} \alpha^{\pm} = \beta \\ E + E_2\beta^2 \approx 0 \\ C_1 + C_2\beta + C_3\beta^2 + C_4\beta^3 \approx 0. \end{gathered} \tag{34}$$

Now, as a condition of the continuity of quasilinear model to the semilinear one, we shall ask that

$$\left.\frac{\partial f(\varepsilon, \tau(\tau_1))}{\partial \tau_1}\right|_{\tau_1 = h(\varepsilon)} = 1. \tag{35}$$

This condition, for the third approximation, can be expressed, using (30), as

$$\left.\frac{\partial f_3}{\partial \tau}\frac{\partial \tau}{\partial \tau_1}\right|_{\tau = l_3(\varepsilon)} = 1 + \left.\frac{C_2\varepsilon + 2C_3\varepsilon^{1/2}\tau + 3C_4\tau^2}{1 + 2E_2\tau}\right|_{\tau = l_3(\varepsilon)} = 1$$

which leads to the equation

$$C_2\varepsilon + 2C_3\varepsilon^{1/2}\tau + 3C_4\tau^2 = 0. \tag{36}$$

From (32) we get

$$3C_1\varepsilon + 2C_2\varepsilon^{1/2}\tau + C_3\tau^2 = 0. \tag{37}$$

These equations must have at least a common root. But our purpose is not to discuss all possibilities. We shall only consider the case when (36) and (37) have two common roots. In order to accomplish this, the following relations have to be satisfied

$$C_2 = 3\lambda C_1, \qquad C_3 = 3\lambda^2 C_1, \qquad C_4 = \lambda^3 C_1 \tag{38}$$

where λ and C_1 remain as unknown constants.

Using (38) in (36) (or 37), and (33), we obtain

$$\tau = -\frac{1}{\lambda}\varepsilon^{1/2} = l_3(\varepsilon) \tag{39}$$

$$\sigma = \left(E + \frac{1}{\lambda^2}E_2\right)\varepsilon - \frac{1}{\lambda}\varepsilon^{1/2} = g_3(\varepsilon). \tag{40}$$

Relaxation boundary (40) can approximate the parabola $\sigma = \beta\varepsilon^{1/2}$ if we choose

$$\lambda = -\frac{1}{\beta} \tag{41}$$

and

$$E + \beta^2 E_2 \approx 0. \tag{42}$$

With the choice in (41) and (42), $E_2 < 0$ and $\tau > 0$; and (30) gives a one-to-one correspondence between τ and τ_1

$$\tau = \frac{-1 + (1 + 4E_2\tau_1)^{1/2}}{2E_2} \tag{30'}$$

if τ_1 is negative. From (30), (39), (41), (42) it follows that $\beta\sqrt{\varepsilon} - E\varepsilon = \tau_1$ and τ_1 is negative for $\varepsilon > (\beta/E)^2 = \varepsilon_y$. The strain ε_y can be obtained as the intersection between the elastic line $\sigma = E\varepsilon$ and the curve $\sigma = \beta\varepsilon^{1/2}$, denoted ε_{y1} in[19]. The numerical results obtained there using this strain as strain at yield point, are not in very good agreement with experimental data, but there $\sigma = \beta\varepsilon^{1/2}$ was used as a finite constitutive equation.

Therefore, the curves $\sigma = \beta\sqrt{\varepsilon}$ and $\sigma = g_3(\varepsilon)$ from (40) could be identified only for $\varepsilon > \varepsilon_y$. In summary, for ($b_3$):

$$\left.\begin{aligned}
&\sigma = g_3(\varepsilon) = \beta\sqrt{\varepsilon} + 0(\varepsilon)\\
&\tau = l_3(\varepsilon) = \beta\sqrt{\varepsilon}\\
&\tau_1 = h_3(\varepsilon) = \tau - (E/\beta^2)\tau^2 = \beta\sqrt{\varepsilon} - E\varepsilon\\
&\sigma = f_3(\varepsilon, 0) = i_3(\varepsilon) = E\varepsilon + C_1\varepsilon^{3/2}\\
&\sigma = f_3(\varepsilon, \tau) = E\varepsilon + \tau - (E/\beta^2)\tau^2 + (C_1/\beta^3)(\beta\sqrt{\varepsilon} - \tau)^3.
\end{aligned}\right\} \tag{43}$$

The term $0(\varepsilon)$ in the equation of the relaxation boundary depends upon the approximation taken in (42) and would be zero for $E + \beta^2 E_2 = 0$. Equation $(43)_4$ gives the instantaneous response curve, while $(43)_5$ gives the actual response curve for $\tau(t)$ determined by a non-instantaneous $\varepsilon(t)$. The condition that τ_1 be negative follows from the fact ψ is negative while $\varepsilon_0 = 0$, $\sigma_0 = 0$, and implies that, for any history τ, $f_3(\varepsilon, \tau) \leq i_3(\varepsilon)$.

(b_4) Consider now the fourth approximation

$$\sigma = f_4(\varepsilon, \tau) = E\varepsilon + \tau + E_2\tau^2 + C_1\varepsilon^{3/2} + C_2\varepsilon\tau + C_3\varepsilon^{1/2}\tau^2 + C_4\tau^3 + D_1\varepsilon^2 + D_2\varepsilon^{3/2}\tau + D_3\varepsilon\tau^2 + D_4\varepsilon^{1/2}\tau^3 + D_5\tau^4. \tag{44}$$

We find, for $\tau = l_4(\varepsilon)$, the following equation

$$D_4\tau^3 + (C_3 + 2D_3\varepsilon^{1/2})\tau^2 + (2C_3\varepsilon^{1/2} + 3D_2\varepsilon)\tau + (3C_1\varepsilon + 4D_1\varepsilon^{3/2}) = 0. \tag{45}$$

Condition (35) leads to the equation

$$4D_5\tau^3 + (3C_4 + 3D_4\varepsilon^{1/2})\tau^2 + (2C_3\varepsilon^{1/2} + 2D_3\varepsilon)\tau + C_2\varepsilon + D_2\varepsilon^{3/2} = 0, \tag{46}$$

which must be satisfied on the relaxation boundary.

Equations (45) and (46) must have at least a common root. If we ask that these equations possess the same roots, then we get $g_4(\varepsilon) = g_3(\varepsilon)$ and $l_4(\varepsilon) = l_3(\varepsilon)$, and if we make the identifications (41) and (42), we obtain the modifications only in the expression for the instantaneous curve and in the equation describing the whole process. These become

$$\sigma = f_4(\varepsilon, 0) = i_4(\varepsilon) = E\varepsilon + C_1\varepsilon^{3/2} + D_1\varepsilon^2$$

$$\sigma = f_4(\varepsilon, \tau) = E\varepsilon + \tau - \frac{E}{\beta^2}\tau^2 + \frac{C_1}{\beta^3}(\beta\sqrt{\varepsilon} - \tau)^3 + \frac{D_1}{\beta^4}(\beta\sqrt{\varepsilon} - \tau)^4. \tag{47}$$

Generally speaking, the instantaneous curve $\sigma = f(\varepsilon, 0) = i(\varepsilon)$ controls the leading edge of the wave propagation in an undeformed material. In this way the fourth approximation might give a better description of the instantaneous curve for the same relaxation boundary.

(c) The foregoing development was made for the special case that the initial state was $\varepsilon_0 = 0$, $\sigma_0 = 0$. We reconsider now the series expansion of part (a) of this section starting from a stable point of nonzero $(\varepsilon_0, \sigma_0)$, i.e. from a state that would maintain $\varepsilon = \varepsilon_0 =$ constant and $\sigma = \sigma_0$ constant until additional loading or unloading is applied. For a new loading from this point,

$$\sigma = f(\varepsilon, \tau, \varepsilon_0) = \sigma_0 + E(\varepsilon - \varepsilon_0) + \tau - \tau_0 + B_1(\varepsilon - \varepsilon_0)^2 + B_2(\varepsilon - \varepsilon_0)(\tau - \tau_0) + B_3(\tau - \tau_0)^2 + C_1(\varepsilon - \varepsilon_0)^3 + C_2(\varepsilon - \varepsilon_0)^2(\tau - \tau_0) + C_3(\varepsilon - \varepsilon_0)(\tau - \tau_0)^2 + C_4(\tau - \tau_0)^3 + \cdots. \tag{48}$$

Here the coefficients B_i, C_i could be different from the coefficients B_i, C_i of (15).

We apply both conditions (8) and (35) to successive polynomial approximations $f_i(\varepsilon, \tau)$. The condition (35), in this case, can be written as

$$\left.\frac{\partial f}{\partial \tau}(\varepsilon, \tau, \varepsilon_0)\right|_{\tau = h(\varepsilon)} = 1. \tag{49}$$

For

$$\sigma = f_2(\varepsilon, \tau) = \sigma_0 + E(\varepsilon - \varepsilon_0) + \tau - \tau_0 + B_1(\varepsilon - \varepsilon_0)^2 + B_2(\varepsilon - \varepsilon_0)(\tau - \tau_0) + B_3(\tau - \tau_0)^2, \tag{50}$$

we find the following relations

$$\begin{aligned} 2B_1(\varepsilon-\varepsilon_0)+B_2(\tau-\tau_0)&=0 \\ B_2(\varepsilon-\varepsilon_0)+2B_3(\tau-\tau_0)&=0 \end{aligned} \qquad t \geq t_2 \tag{51}$$

imposed by conditions (8) and (49).

These conditions must hold for any ε close to ε_0 ($\varepsilon > \varepsilon_0$), and therefore we have

$$B_2{}^2 = 4B_1B_3 \qquad \text{or} \qquad B_2 = 2\lambda B_1, \qquad B_3 = \lambda^2 B_1. \tag{52}$$

Combining $(51)_1$ with (50) and taking into account (52), we get the equation for the relaxation boundary as

$$\sigma = g_2(\varepsilon) = \sigma_0 + E_1(\varepsilon-\varepsilon_0), \qquad E_1 = E - \frac{1}{\lambda}. \tag{53}$$

The instantaneous curve is obtained for $\tau = \tau_0$, and it has the equation

$$\sigma = f_2(\varepsilon, \tau_0) = i_2(\varepsilon) = \sigma_0 + E(\varepsilon-\varepsilon_0) + B_1(\varepsilon-\varepsilon_0)^2. \tag{54}$$

The corresponding approximation for the function ϕ is obtained by using the relations

$$\begin{aligned} &\frac{\partial f_2}{\partial \varepsilon}(\varepsilon, \tau) = \phi(\varepsilon, f_2(\varepsilon, \tau)) \\ &\tau - \tau_0 = \frac{-[1+B_2(\varepsilon-\varepsilon_0)] + \sqrt{[[1+B_2(\varepsilon-\varepsilon_0)]^2 - 4B_3(i_2(\varepsilon)-\sigma)]}}{2B_3} \end{aligned} \tag{55}$$

and has the expression

$$\phi_2(\varepsilon, \sigma) = E + (E - E_1)(\sqrt{[1 + 4B_3[\sigma - \sigma_0 - E_1(\varepsilon-\varepsilon_0)]]} - 1)]. \tag{56}$$

Therefore if E is known, in order to determine the second approximation of function ϕ we need to know two constants, the slope of the relaxation boundary E_1 and a constant B_i ($i = 1, 2, 3$) related to the instantaneous response.

We have chosen the sign plus in $(55)_2$ based on the following two arguments: (1) when over the state (ε_0, σ_0) is applied a jump in strain, the stress has to follow the instantaneous curve (i.e. τ has to be equal to τ_0), and (2) for ε close enough to ε_0, $1 + B_2(\varepsilon - \varepsilon_0) > 0$.

Now, if we suppose, as usual, that $\psi(\varepsilon, \sigma) \leq 0$, then from (5) it follows that $\tau < \tau_0$, and from (51), (52) and (53), we obtain

$$\lambda > 0, \qquad E > E_1 \tag{57}$$

and the coefficients B_1, B_2, B_3 have the same sign. If these coefficients are negative, then some additional restrictions on the domains of functions $h_2(\varepsilon)$ and $\phi_2(\varepsilon, \sigma)$ are implied. If these coefficients are positive, such restrictions are not implied, but the instantaneous curve is concave toward the σ-axis, and the predicted actual speed of propagation would be larger than or equal to bar velocity C_0.

5. CONCLUSIONS AND REMARKS

The results of this paper are based on several hypotheses. Constitutive assumption II is made because all experimental evidence indicates that the leading edge of an incremental wave travels at the elastic bar wave speed. One form or another of constitutive assumption I has been accepted in many fields of physics, at least locally.

The constitutive assumptions I and II and the further hypothesis that $\partial\phi/\partial\sigma \neq 0$, were sufficient conditions to obtain theorem 3.1, showing that (at least locally) there exists a unique relaxation boundary, i.e. there exists in the ε, σ-plane a curve $\sigma = g(\varepsilon)$ such that $\phi(\varepsilon, g(\varepsilon)) = E$ and $\psi(\varepsilon, g(\varepsilon)) = 0$.

The hypothesis $\partial\phi/\partial\sigma \neq 0$ is sufficient but not necessary for the existence of a unique relaxation boundary. For example, the semilinear equation with $\phi \equiv E$ and $\psi = -k[\sigma - g(\varepsilon)]$ has the unique relaxation boundary $\sigma = g(\varepsilon)$. A weaker hypothesis, that $\partial\phi/\partial\sigma \not\equiv 0$, is sufficient to ensure the existence of a relaxation boundary at least in the regions where $\partial\phi/\partial\sigma \neq 0$.

A relaxation boundary may be introduced by the structure of the function ψ, as in the semilinear example cited.

Constitutive assumption II dealt with an "absolute plateau" $\varepsilon =$ constant before the incremental pulse. Because of the assumed smoothness of all the functions involved, the same results can be expected to a high degree of approximation for an "asymptotic plateau," where the strain is still increasing slowly when the incremental pulse arrives. Similar comments apply to the expansions of section (c) in the neighborhood of a "stable point."

The hypothesis (35) goes beyond constitutive assumption II and gives the semilinear constitutive equation a special status in the class of quasilinear constitutive equations as the limiting form in the neighborhood of any relaxation boundary. This additional hypothesis led to the close connection between the relaxation boundary and the function ϕ, shown in section 4. These results are interesting and open up possibilities of determining ϕ, since the relaxation boundary can be determined experimentally. The physical basis for the hypothesis remains to be determined.

The series expansions of section 4 depend on the point $(\bar{\varepsilon}, \bar{\sigma})$ where $\psi(\varepsilon, \sigma)$ starts to be different from zero. For $\varepsilon < \bar{\varepsilon}$, $\tau = 0$, and the whole process is described by the instantaneous curve up to $(\bar{\varepsilon}, \bar{\sigma})$. Series expansions can be made in the neighborhood of any stable point $\sigma(t) = \sigma_0 =$ constant, $\varepsilon(t) = \varepsilon_0 =$ constant on a relaxation boundary where $\psi(\varepsilon_0, \sigma_0) = 0$, as in section (c). These series expansions show that knowledge of a relaxation boundary actually gives us more information about ϕ than about ψ. In fact, if we could determine all the coefficients of such a series, then ϕ could be completely determined by a procedure like that of section (c).

The way in which the history parameter τ reaches its relaxation boundary $\tau = h(\varepsilon)$ and the time interval $t_2 - t_1$ from constitutive assumption II are determined mainly by the function ψ. The hypothesis that ψ is a linear function of "overstress" appears as a first approximation. But, as can be seen from section 4, the history parameter has large variations with ε, and probably in most metals it has a very fast time variation (short "relaxation time"). Thus a linear approximation may not be adequate. Even the hypothesis that ψ is a function only of overstress may not be justified.

Similar results to those given in this paper can be obtained for the case where we suppose the history parameter τ is a functional of strain history (not necessarily obtained as a solution of equations (2) and (5)) with some smoothness properties (see for example, Coleman, Gurtin and Herrera[21]). Also the results of section 3 do not depend on the fact that C_0 is a constant bar velocity; C_0 could be a positive function depending on the level of prestrain, but differing from the plastic wave speed calculated with a finite stress-strain law.

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Абстракт — Обсуждается условие, когда ведущий край постепенно нарастающего импульса нагрузки, наложен на предварительную нагрузку, которая уже достигнула неизменное положение, распространяется со скоростью упругой волны. Это условие накладывает некоторые ограничения функций материала, описывающих поведение материала типа квазилинейной скорости. Исследуются эти ограничения. Оказывается, что поведение постепенно нарастающей волны означает существование, по крайной мере, одной границы реляксации и, далее, что натурный непреривный переход из квазилинейного уравнения к полулинейному происходит в окрестности границы реляксации.

Рассматривается, также, способность пользования знакомством границы реляксации, в целью определения функций материала.